

Reduction operators and exact solutions of variable coefficient nonlinear wave equations with power nonlinearities

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Reduction operators, i.e. the operators of nonclassical (or conditional) symmetry of a class of variable coefficient nonlinear wave equations containing three arbitrary functions and two power nonlinearities is investigated within the framework of singular reduction operator. A classification of regular reduction operators is performed with respect to generalized extended equivalence groups. Exact solutions of some nonlinear wave model which are invariant under certain reduction operators are also constructed.

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1 Introduction

The investigation of Lie-point symmetry of differential equations is a well established and powerful tool for constructing exact solutions of the equations under consideration [6, 19, 27, 29]. In fact, classical Lie-point symmetry group analysis is the only systematic method we know for deducing exact solutions (in particular, invariant, partially invariant and separation of variables solutions) of nonlinear partial differential equations [6, 19, 22, 26, 27, 29]. However, for a lot of important applications differential equations because their classical Lie symmetry groups are rather trivial including at most space and time translations and scale transformations, the obtained solutions are not sufficient for our understanding the mechanics and mathematics of mathematical models. Thus, this stimulates the efforts devoted to generalization of Lie's original concept of symmetry in order to find more other types exact solutions.

The first approach to these symmetries is conditional ones (called also nonclassical symmetries or Q -conditional symmetries), which was introduced by Bluman and Cole [5] in 1969. It consists in augmenting the original partial differential equations with invariant surface conditions, a system of first order differential equations satisfied by all functions invariant under a certain vector field. The latter is chosen as a classical infinitesimal symmetry for the augmented system. The basic equations for conditional symmetries are similar to Lie's determining equations except that they are nonlinear and less overdetermined. That is why one rarely succeeds in obtaining all possible solutions to the determining equations for conditional symmetries, especially in the case of second order equations.

In contrast to classical Lie symmetry, another difference in the research of such two kinds of

symmetries is the procedure of deriving the determining equations. Namely, deriving systems of determining equations for conditional symmetries crucially depends on the interplay between the operators and the equations under consideration. For example, for the linear heat equation $u_t = u_{xx}$, the determining equations are derived usually based on the condition that the general form of conditional symmetry operators $Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$, where $(\tau, \xi) \neq (0, 0)$, were divided into two essentially different cases, i.e., the regular case $\tau \neq 0$ and the singular case $\tau = 0$. However, such partition of sets of conditional symmetry operators with the conditions of vanishing and nonvanishing coefficients of operators is not appropriate for all of differential equations. Therefore, investigation of algorithms for deriving the determining equations for condition symmetries are essential importance, which will provide some useful guidelines for simplifying the obtained determining equation and make its complete solving become possible. Recently, Popovych et.al give a detail investigation on this question and present a novel framework, namely singular reduction operators or singular reduction modules [8, 24], for finding an optimal way of obtaining the determining equation of conditional symmetries. As examples, properties of singular reduction operators of (1+1)-dimensional evolution equations and a specific wave equations have been studied [24, 30–32, 39, 41]. However, for more general nonlinear wave equation, there exist no general results.

In this paper we extend this new framework of singular reduction operators [24] to study a class of variable coefficient nonlinear wave equations which possesses three arbitrary functions $f = f(x)$, $g = g(x)$ and $h = h(x)$ and two power nonlinearities u^n and u^m of the form

$$f(x)u_{tt} = (g(x)u^n u_x)_x + h(x)u^m, \quad (1)$$

where $f(x)g(x) \neq 0$, n and m are arbitrary constants, t is the time coordinate and x is the one-space coordinate. The linear case is excluded from consideration because it was well-investigated. We also assume the variable wave speed coefficient u^n to be nonlinear, i.e. $n \neq 0$. The case $n = 0$ is quite singular and will be investigated separately.

Many specific nonlinear wave models describing a wide variety of phenomena in Mechanics and Engineering such as the flow of one-dimensional gas, shallow water waves theory, longitudinal wave propagation on a moving threadline, dynamics of a finite nonlinear string, elastic-plastic materials and electromagnetic transmission line and so on, can be reduced to equation (1) (see [1] p.50-52 and [2]). Classical Lie symmetries of various kinds of quasi-linear wave equations in two independent variables that intersect class (1) have been investigated in [3, 4, 6, 7, 9, 10, 12–14, 17–20, 28, 35–37, 42]. Recently, we have present a complete local symmetry and conservation law classification of class (1) in [15, 16]. Classical Lie symmetry reduction and invariant solutions of some variable coefficients wave models which are singled out from the classification results are also investigated. However, the nonclassical symmetries of class (1) remains open. In this paper, we will make a detailed investigation on this subjects so that we can use it to obtain some new non-Lie exact solutions. Below, following [24] we use the shorter and more natural term ‘reduction operators’ instead of ‘operators of conditional symmetry’ or ‘operators of nonclassical symmetry’.

The rest of paper is organized as follows. In Section 2, known results on equivalence transformations and classical group analysis of the wave equations (1) are adduced in a form which is suitable for purposes of our investigations. Then in section 3, singular reduction operators, and in particular regular reduction operators classification for the class under consideration are investigated after some brief review of the notation of singular and regular reduction operator. Section 4 contains nonclassical symmetry reduction of some nonlinear wave models including some *truly* ‘variable coefficient’ ones which are singled out from the class of (1+1)-dimensional wave equations. New non-Lie exact solutions of the models are constructed by means the reduction. Conclusions and discussion are given in section 5.

2 Equivalence transformations and Lie symmetries

The exhaustive result on classical group classification of class (1) is presented by the statements adduced below [15]. Using the direct method, we construct different kinds of equivalence groups including usual and generalized extended ones and discuss their structure (for the details about equivalence group can be seen in [21, 29, 33, 34, 38]).

Theorem 1. *The usual equivalence group G^\sim for the class (1) consists of the transformations*

$$\begin{aligned}\tilde{t} &= \epsilon_2 t + \epsilon_3, & \tilde{x} &= X(x), & \tilde{u} &= \epsilon_4 u, \\ \tilde{f} &= \epsilon_1 \epsilon_2^2 \epsilon_4^{-1} X_x^{-1} f, & \tilde{g} &= \epsilon_1 \epsilon_4^{-n-1} X_x g, & \tilde{h} &= \epsilon_1 \epsilon_4^{-m} X_x^{-1} h, & \tilde{n} &= n, & \tilde{m} &= m,\end{aligned}$$

where ϵ_j ($j = 1, \dots, 4$) are arbitrary constants, $\epsilon_1 \epsilon_2 \epsilon_4 \neq 0$, X is an arbitrary smooth function of x , $X_x \neq 0$.

It is shown that class (1) admits other equivalence transformations which do not belong to G^\sim and form, together with usual equivalence transformations, an *generalized extended equivalence group*. Using the direct method [23], we can construct the following complete generalized extended equivalence group \hat{G}^\sim of class (1).

Theorem 2. *The generalized extended equivalence group \hat{G}^\sim of the class (1) is formed by the transformations*

$$\begin{aligned}\tilde{t} &= \epsilon_2 t + \epsilon_3, & \tilde{x} &= X(x), & \tilde{u} &= Y(x)u, \\ \tilde{f} &= \epsilon_1 \epsilon_2^2 X_x^{-1} Y^{-n-2} f, & \tilde{g} &= \epsilon_0 X_x Y^{-2n-2} g, & \tilde{h} &= \epsilon_1 X_x^{-1} Y^{-m-n-1} h, & \tilde{n} &= n, & \tilde{m} &= m,\end{aligned}$$

where ϵ_j ($j = 1, 2, 3$) are arbitrary constants, $\epsilon_1 \epsilon_2 \neq 0$, X is an arbitrary smooth function of x , $X_x \neq 0$, the function $Y(x)$ is determined by the formula

$$Y(x) = \begin{cases} (1 - (n+1)F(x))^{-\frac{1}{n+1}}, & n \neq -1, \\ e^{F(x)}, & n = -1 \end{cases}, \quad \text{where} \quad F(x) = \epsilon_4 \int \frac{dx}{g(x)} + \epsilon_5.$$

Using the transformation

$$\tilde{t} = t, \quad \tilde{x} = \int \frac{dx}{g(x)}, \quad \tilde{u} = u \tag{2}$$

from theorem 1, we can reduce equation (1) to $\tilde{f}(\tilde{x})\tilde{u}_{\tilde{t}\tilde{t}} = (\tilde{u}^n \tilde{u}_{\tilde{x}})_{\tilde{x}} + \tilde{h}(\tilde{x})\tilde{u}^m$, where $\tilde{f}(\tilde{x}) = g(x)f(x)$, $\tilde{g}(\tilde{x}) = 1$ and $\tilde{h}(\tilde{x}) = g(x)h(x)$. (Similarly, any equation of form (1) can be reduced to the same form with $\tilde{f}(\tilde{x}) = 1$.) Thus, without loss of generality we can restrict ourselves to investigation of the equation

$$f(x)u_{tt} = (u^n u_x)_x + h(x)u^m. \tag{3}$$

All results on symmetries, solutions and conservation laws of class (3) can be extended to class (1) with transformations (2).

Theorem 3. *The usual equivalence group G_1^\sim for the class (3) with $n \neq -1$ is formed by the transformations*

$$\begin{aligned}\tilde{t} &= \epsilon_1 t + \epsilon_2, & \tilde{x} &= \frac{1}{\delta(n+1)\epsilon_3[1-(n+1)(\epsilon_3 x + \epsilon_4)]} - \frac{1}{\delta(n+1)\epsilon_3} =: X(x), \\ \tilde{u} &= [1 - (n+1)(\epsilon_3 x + \epsilon_4)]^{-\frac{1}{n+1}} u = (\delta X_x)^{\frac{1}{2n+2}} u =: Y(x)u, \\ \tilde{f} &= \epsilon_1^2 \delta^{\frac{n}{2n+2}} X_x^{-\frac{3n+4}{2n+2}} f, & \tilde{h} &= \delta^{\frac{-m+n+1}{2n+2}} X_x^{-\frac{m+3n+3}{2n+2}} h, & \tilde{n} &= n, & \tilde{m} &= m,\end{aligned}$$

where ϵ_j ($j = 1, \dots, 4$) and δ are arbitrary constants, $\delta \epsilon_1 \epsilon_3 \neq 0$.

The above representation of the transformations from G_1^\sim is very clumsily, but it shows that the family of transformations is continuously parameterized with the parameter n , including the value $n = -1$. In fact, we have $\lim_{n \rightarrow -1} X(x) = \frac{x}{\delta} + \frac{\epsilon_4}{\delta \epsilon_3}$, $\lim_{n \rightarrow -1} Y(x) = e^{\epsilon_3 x + \epsilon_4}$. Therefore, we can reformulate the above transformations as a more simple and clear formulae than in theorem 3.

Theorem 4. *The generalized equivalence group G_1^\sim for the class (3) with $n \neq -1$ consists of the transformations*

$$\begin{aligned} \tilde{t} &= \epsilon_1 t + \epsilon_2, \quad \tilde{x} = \frac{\epsilon_3 x + \epsilon_4}{\epsilon_5 x + \epsilon_6} =: X(x), \quad \tilde{u} = \epsilon_7 X_x^{\frac{1}{2n+2}} u, \\ \tilde{f} &= \epsilon_1^2 \epsilon_7^n X_x^{-\frac{3n+4}{2n+2}} f, \quad \tilde{h} = \epsilon_7^{-m+n+1} X_x^{-\frac{m+3n+3}{2n+2}} h, \quad \tilde{n} = n, \quad \tilde{m} = m, \end{aligned}$$

where ϵ_j ($j = 1, \dots, 7$) are arbitrary constants, $\epsilon_1 \epsilon_7 \neq 0$ and $\epsilon_3 \epsilon_6 - \epsilon_4 \epsilon_5 = \pm 1$.

For $n = -1$ transformations from group G_1^\sim takes the form

$$\begin{aligned} \tilde{t} &= \epsilon_1 t + \epsilon_2, \quad \tilde{x} = \epsilon_3 x + \epsilon_4, \quad \tilde{u} = \epsilon_5 e^{\epsilon_6 x} u, \\ \tilde{f} &= \epsilon_1^2 \epsilon_3^{-2} \epsilon_5^{-1} e^{-\epsilon_6 x} f, \quad \tilde{h} = \epsilon_3^{-2} \epsilon_5^{-m} e^{-m \epsilon_6 x} h, \quad \tilde{n} = n, \quad \tilde{m} = m, \end{aligned}$$

where ϵ_j ($j = 1, \dots, 6$) are arbitrary constants, $\epsilon_1 \epsilon_3 \epsilon_5 \neq 0$.

Since the parameters n and m are invariants of all the above equivalence transformations, class (1) (or class (3)) can also be presented as the union of disjoint subclasses where each from the subclasses corresponds to fixed values of n and m . Thus the generalized equivalence group \hat{G}^\sim of class (1) can be considered as a family of usual equivalence groups of the subclasses parameterized with n and m . Motivated by this, we can obtain wider equivalence groups for some of the subclasses apriori assuming the parameters n and m satisfy a condition. In particular, for the case $m = n + 1$, we have

Theorem 5. *The class of equations*

$$f(x)u_{tt} = (g(x)u^n u_x)_x + h(x)u^{n+1} \quad (4)$$

admits the equivalence transformation group $G_{m=n+1}^\sim$ consisting of the transformations:

$$\begin{aligned} \tilde{t} &= \epsilon_2 t + \epsilon_3, \quad \tilde{x} = X(x), \quad \tilde{u} = Y(x)u, \quad \tilde{f} = \epsilon_1 \epsilon_2^2 X_x^{-1} Y^{-n-2} f, \\ \tilde{g} &= \epsilon_1 X_x Y^{-2n-2} g, \quad \tilde{h} = \epsilon_1 [h - Y^{n+1} (Y^{-(n+2)} Y_x g)_x] X_x^{-1} Y^{-2n-2} h, \quad (\tilde{n} = n), \end{aligned}$$

where X and Y are arbitrary functions of x , ϵ_j ($j = 1, 2, 3$) are arbitrary constants, $\epsilon_1 \epsilon_2 X_x Y \neq 0$.

If we further gauge g with the condition $g = 1$ and impose the restrictions $\epsilon_1 X_x = Y^{2n+2}$ on parameters of $G_{m=n+1}^\sim$, we can obtain the following equivalence group $G_{1,m=n+1}^\sim$ for the class of equations

$$f(x)u_{tt} = (u^n u_x)_x + h(x)u^{n+1}. \quad (5)$$

Corollary 1. *The equivalence group $G_{1,m=n+1}^\sim$ of class (5) is formed by the transformations:*

$$\begin{aligned} \tilde{t} &= \epsilon_2 t + \epsilon_3, \quad \tilde{x} = X(x), \quad \tilde{u} = Y(x)u, \\ \tilde{f} &= \epsilon_1^2 \epsilon_2^2 Y^{-3n-4} f, \quad \tilde{h} = \epsilon_1^2 [h - Y^{n+1} (Y^{-(n+2)} Y_x)_x] Y^{-4n-4} h, \quad (\tilde{n} = n), \end{aligned}$$

where X and Y are arbitrary functions satisfying the condition $\epsilon_1 X_x = Y^{2n+2}$, ϵ_j ($j = 1, 2, 3$) are arbitrary constants, $\epsilon_1 \epsilon_2 Y \neq 0$.

Using the above-mentioned results, we obtained the classical symmetry classification for the class (3) in [15].

In table 1, we list tuples of parameter-functions $f(x)$ and $h(x)$, the constant parameters m and n and bases of the invariance algebras in all possible inequivalent cases of Lie symmetry extension. The operators from table 1 form bases of the maximal Lie invariance algebras iff the corresponding values of the parameters are inequivalent to ones with more abundant Lie invariance algebras. In cases 2 and 3 we do not try to use equivalence transformations as much as possible since otherwise a number of similar simplified cases would be derived, see footnote after the table.

Table 1. Results of group classification of class (3)

N	n	$f(x)$	$h(x)$	Basis of A^{\max}
General case				
1	\forall	\forall	\forall	$\partial_t,$
2	\forall	$f^1(x)$	$h^1(x)$	$\partial_t, \frac{1}{2}(d+2b-pn)t\partial_t + [a(n+1)x^2 + bx + c]\partial_x$ $+(ax+p)u\partial_u,$
3	\forall	1	ε	$\partial_t, \partial_x, (1-m)t\partial_t + (1+n-m)x\partial_x + 2u\partial_u$
$m=1, h \neq 0, (h/f)_x = 0$				
4	$\neq -4$	$f^1(x)$	εf	$\partial_t, n[a(n+1)x^2 + bx + c]\partial_x + (nax + 2b + d)u\partial_u$
5	$\neq -4, -\frac{4}{3}$	1	ε	$\partial_t, \partial_x, nx\partial_t + 2u\partial_u$
6	-4	$f^1(x) _{n=-4}$	εf	$\partial_t, e^{2t}(\partial_t + u\partial_u), e^{-2t}(\partial_t - u\partial_u), 4(-3ax^2 + bx + c)\partial_x$ $+(4ax - d - 2b)u\partial_u$
7	$-\frac{4}{3}$	1	ε	$\partial_t, \partial_x, 2x\partial_x - 3u\partial_u, x^2\partial_x - 3xu\partial_u$
8	-4	1	1	$\partial_t, \partial_x, e^{2t}(\partial_t + u\partial_u), 2x\partial_x - u\partial_u, e^{-2t}(\partial_t - u\partial_u)$
9	-4	1	-1	$\partial_t, \partial_x, \cos(2t)\partial_t - \sin(2t)u\partial_u, 2x\partial_x - u\partial_u,$ $\sin(2t)\partial_t + \cos(2t)u\partial_u$
$m=n+1$ or $h=0$				
10	$\neq -4$	\forall	\forall	$\partial_t, nt\partial_t - 2u\partial_u$
11	$\neq -4, -\frac{4}{3}$	1	αx^{-2}	$\partial_t, t\partial_t + x\partial_x, nt\partial_t - 2u\partial_u,$
12	$\neq -4, -\frac{4}{3}$	1	ε	$\partial_t, \partial_x, nt\partial_t - 2u\partial_u$
13	$\neq -4, -\frac{4}{3}$	1	0	$\partial_t, \partial_x, t\partial_t + x\partial_x, nt\partial_t - 2u\partial_u$
14	$-\frac{4}{3}$	e^x	α	$\partial_t, t\partial_t + 2\partial_x, 2t\partial_t + 3u\partial_u$
15	$-\frac{4}{3}$	1	0	$\partial_t, \partial_x, t\partial_t + x\partial_x, 2t\partial_t + 3u\partial_u, x^2\partial_x - 3xu\partial_u$
16	-4	1	0	$\partial_t, \partial_x, t\partial_t + x\partial_x, 2t\partial_t + u\partial_u, t^2\partial_x + tu\partial_u$

Here α is arbitrary constant, $\varepsilon = \pm 1$,

$$f^1(x) = \exp \left\{ \int \frac{-(3n+4)ax + d}{(n+1)ax^2 + bx + c} dx \right\}, \quad h^1(x) = \exp \left\{ \int \frac{-(3(n+1)+m)ax + (1+m-n)p - 2b}{(n+1)ax^2 + bx + c} dx \right\},$$

and it can be assumed up to equivalence with respect to G_1^\sim that the parameter tuple (a, b, c, d) takes only the following non-equivalent values:

$$(\varepsilon, 0, 1, 0) \quad \text{if } n = -1 \quad \text{or} \quad (1, 0, 1, d') \quad \text{if } n \neq -1, \quad (0, 1, 0, d'), \quad (0, 0, 1, 1),$$

where d' is arbitrary constant. In all the cases we put $g(x) = 1$. In case 10 the parameter-functions f and h can be additionally gauged with equivalence transformations from $G_{1, m=n+1}^\sim$. For example, we can put $f = 1$ if $n \neq -4, -4/3$ and $f = e^x$ otherwise.

Remark 1. It should be emphasized that we adduce only the cases of extensions of maximal Lie invariance algebra, which are inequivalent with respect to G_1^\sim if $m \neq n+1$ and with respect to $G_{1, m=n+1}^\sim$ if $m = n+1$ or $h = 0$ in table 1. What's more, right choice of a gauge of the parameter tuple (f, g, h) is another crucial point of our investigations. In fact, the major choice have been made from the very outset of classification when the parameter-function g was put equal to 1. It is the gauge that leads to maximal simplification of both the whole solving and

the final results. Although all gauges are equivalent from an abstract point of view, only the gauge $g = 1$ allows one to exhaustively solve the problem of group classification of class (1) with reasonable quantity of calculations. Even after the other simplest gauge $f = 1$ chosen, calculations become too cumbersome and sophisticated.

3 Nonclassical symmetries

In this section, we first review some necessary definitions and statements on nonclassical symmetries [11, 25, 40, 43] and singular reduction operator [24]. Then perform a detailed investigation on the reduction operators of class (3).

Consider an r th order differential equation \mathcal{L} of the form $L(t, x, u_{(r)}) = 0$ for the unknown function u of the two independent variables t and x , where $L = L[u] = L(t, x, u_{(r)})$ is a fixed differential function of order r and $u_{(r)}$ denotes the set of all the derivatives of the function u with respect to t and x of order not greater than r , including u as the derivative of order zero.

In order to discuss the conditional symmetries of equation \mathcal{L} , we will first treat equation \mathcal{L} from a geometric point of view as an algebraic equation in the jet space J^r of order r and is identified with the manifold of its solutions in J^r [27]:

$$\mathcal{L} = \{(t, x, u_{(r)}) \in J^r | L(t, x, u_{(r)}) = 0\}.$$

Let \mathcal{Q} denote the set of vector fields of the general form

$$Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u, \quad (\tau, \xi) \neq (0, 0), \quad (6)$$

which is a first-order differential operator on the space $\mathbb{R}^2 \times \mathbb{R}^1$ with coordinates t, x , and u . Then all functions invariant under Q and only such functions satisfy a first order differential equation

$$Q[u] := \tau u_t + \xi u_x - \eta = 0 \quad (7)$$

called the the characteristic equation (also known as invariant surface condition).

Denote the manifold defined by the set of all the differential consequences of the characteristic equation $Q[u] = 0$ in J^r by $\mathcal{Q}_{(r)}$, i.e.,

$$\mathcal{Q}_{(r)} = \{(t, x, u_{(r)}) \in J^r | D_t^\alpha D_x^\beta Q[u] = 0, \alpha, \beta \in \mathbb{N} \cup \{0\}, \alpha + \beta < r\},$$

where $D_t = \partial_t + u_{\alpha+1, \beta} \partial_{u_{\alpha, \beta}}$ and $D_x = \partial_x + u_{\alpha, \beta+1} \partial_{u_{\alpha, \beta}}$ are the operators of total differentiation with respect to the variables t and x , and the variable $u_{\alpha, \beta}$ of the jet space J^r corresponds to the derivative $\frac{\partial^{\alpha+\beta} u}{\partial t^\alpha \partial x^\beta}$.

Denote also by $Q_{(r)}$ the standard r th prolongation of Q to the space J^r :

$$Q_{(r)} = Q + \sum_{0 < \alpha + \beta \leq r} \eta^{\alpha\beta} \partial_{u_{\alpha, \beta}}, \quad \eta^{\alpha\beta} := D_t^\alpha D_x^\beta Q[u] + \tau u_{\alpha+1, \beta} + \xi u_{\alpha, \beta+1}.$$

Definition 1. The differential equation \mathcal{L} is called conditionally invariant with respect to the operator Q if the relation

$$Q_{(r)}[L(t, x, u_{(r)})] \Big|_{\mathcal{L} \cap \mathcal{Q}_{(r)}} = 0 \quad (8)$$

holds, which is called the conditional invariance criterion. Then Q is called conditional symmetry (or nonclassical symmetry, Q -conditional symmetries or reduction operator) of the equation \mathcal{L} .

We denote the set of reduction operators of the equation \mathcal{L} by $\mathcal{Q}(\mathcal{L})$ which is a subset of \mathcal{Q} . Any Lie symmetry operator of \mathcal{L} belongs to $\mathcal{Q}(\mathcal{L})$. Sometimes $\mathcal{Q}(\mathcal{L})$ is exhausted by the operators equivalent to Lie symmetry ones in the sense of the following definition.

Definition 2. Two differential operators Q and \tilde{Q} in \mathcal{Q} are called equivalent ($Q \sim \tilde{Q}$) if they differ by a multiplier which is a non-vanishing function of t, x and u : $\tilde{Q} = \lambda Q$, where $\lambda = \lambda(t, x, u)$, $\lambda \neq 0$.

Consider a vector field Q in the form (6) and a differential function $L = L[u]$ of order $\text{ord } L = r$ (i.e., a smooth function of variables t, x, u and derivatives of u of orders up to r).

Definition 3. The vector field Q is called *singular* for the differential function L if there exists a differential function $\tilde{L} = \tilde{L}[u]$ of an order less than r such that $L|_{\mathcal{Q}_{(r)}} = \tilde{L}|_{\mathcal{Q}_{(r)}}$. Otherwise Q is called a *regular* vector field for the differential function L . If the minimal order of differential functions whose restrictions on $\mathcal{Q}_{(r)}$ coincide with $L|_{\mathcal{Q}_{(r)}}$ equals k ($k < r$) then the vector field Q is said to be of *singularity co-order* k for the differential function L . The vector field Q is called *ultra-singular* for the differential function L if $L|_{\mathcal{Q}_{(r)}} \equiv 0$.

Definition 4. A vector field Q is called *weakly singular* for the differential equation $\mathcal{L} : L[u] = 0$ if there exists a differential function $\tilde{L} = \tilde{L}[u]$ of an order less than r and a nonvanishing differential function $\lambda = \lambda[u]$ of an order not greater than r such that $L|_{\mathcal{Q}_{(r)}} = \lambda \tilde{L}|_{\mathcal{Q}_{(r)}}$. Otherwise Q is called a *weakly regular* vector field for the differential equation \mathcal{L} . If the minimal order of differential functions whose restrictions on $\mathcal{Q}_{(r)}$ coincide, up to nonvanishing functional multipliers, with $L|_{\mathcal{Q}_{(r)}}$ is equal to k ($k < r$) then the vector field Q is said to be *weakly singular of co-order* k for the differential equation \mathcal{L} .

Definition 5. A vector field Q is called a *singular reduction operator* of a differential equation \mathcal{L} if Q is both a reduction operator of \mathcal{L} and a weakly singular vector field of \mathcal{L} .

After the factorization of the reduction operator under the usual equivalence relation of reduction operators in Definition 2 to singular and regular cases, the classification of reduction operators can be considerably enhanced and simplified by considering Lie symmetry and equivalence transformations of (classes of) equations.

Lemma 1. Any point transformation of t, x and u induces a one-to-one mapping of \mathcal{Q} into itself. Namely, the transformation $g : \tilde{t} = T(t, x, u)$, $\tilde{x} = X(t, x, u)$, $\tilde{u} = U(t, x, u)$ generates the mapping $g_* : \mathcal{Q} \rightarrow \mathcal{Q}$ such that the operator Q is mapped to the operator $g_*Q = \tilde{\tau}\partial_{\tilde{t}} + \xi\partial_{\tilde{x}} + \eta\partial_{\tilde{u}}$, where $\tilde{\tau}(\tilde{t}, \tilde{x}, \tilde{u}) = QT(t, x, u)$, $\xi(\tilde{t}, \tilde{x}, \tilde{u}) = QX(t, x, u)$, $\eta(\tilde{t}, \tilde{x}, \tilde{u}) = QU(t, x, u)$. If $Q' \sim Q$ then $g_*Q' \sim g_*Q$. Therefore, the corresponding factorized mapping $g_f : \mathcal{Q}_f \rightarrow \mathcal{Q}_f$ also is well defined and bijective.

Lemma 1 results in appearing equivalence relation between operators, which differs from usual one described in Definition 2.

Definition 6. Two differential operators Q and \tilde{Q} in \mathcal{Q} are called equivalent with respect to a group G of point transformations if there exists $g \in G$ for which the operators Q and $g_*\tilde{Q}$ are equivalent. We denote this equivalence by $Q \sim \tilde{Q} \text{ mod } G$.

The problem of finding reduction operators is more complicated than the similar problem for Lie symmetries because the first problem is reduced to the integration of an overdetermined system of nonlinear PDEs, whereas in the case of Lie symmetries one deals with a more overdetermined system of linear PDEs. The question occurs: could we use equivalence and gauging transformations in investigation of reduction operators as we do for finding Lie symmetries? The following statements give the positive answer.

Lemma 2. *Given any point transformation g of an equation \mathcal{L} to an equation $\tilde{\mathcal{L}}$, g_* maps $\mathcal{Q}(\mathcal{L})$ to $\mathcal{Q}(\tilde{\mathcal{L}})$ bijectively. The same is true for the factorized mapping g_f from $\mathcal{Q}_f(\mathcal{L})$ to $\mathcal{Q}_f(\tilde{\mathcal{L}})$.*

Corollary 2. *Let G be the point symmetry group of an equation \mathcal{L} . Then the equivalence of operators with respect to the group G generates equivalence relations in $\mathcal{Q}(\mathcal{L})$ and in $\mathcal{Q}_f(\mathcal{L})$.*

Consider the class $\mathcal{L}|_{\mathcal{S}}$ of equations \mathcal{L}_{θ} : $L(t, x, u_{(r)}, \theta) = 0$ parameterized with the parameter-functions $\theta = \theta(t, x, u_{(r)})$. Here L is a fixed function of $t, x, u_{(r)}$ and θ . The symbol θ denotes the tuple of arbitrary (parametric) differential functions $\theta(t, x, u_{(r)}) = (\theta^1(t, x, u_{(r)}), \dots, \theta^k(t, x, u_{(r)}))$ running through the set \mathcal{S} of solutions of the system $S(t, x, u_{(r)}, \theta_{(q)}(t, x, u_{(r)})) = 0$. This system consists of differential equations on θ , where t, x and $u_{(r)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the derivatives of θ of order not greater than q . In what follows we call the functions θ arbitrary elements. Denote the point transformation group preserving the form of the equations from $\mathcal{L}|_{\mathcal{S}}$ by G^{\sim} .

Let P denote the set of the pairs consisting of an equation \mathcal{L}_{θ} from $\mathcal{L}|_{\mathcal{S}}$ and an operator Q from $\mathcal{Q}(\mathcal{L}_{\theta})$. In view of Lemma 2, the action of transformations from the equivalence group G^{\sim} on $\mathcal{L}|_{\mathcal{S}}$ and $\{\mathcal{Q}(\mathcal{L}_{\theta}) | \theta \in \mathcal{S}\}$ together with the pure equivalence relation of differential operators naturally generates an equivalence relation on P .

Definition 7. Let $\theta, \theta' \in \mathcal{S}$, $Q \in \mathcal{Q}(\mathcal{L}_{\theta})$, $Q' \in \mathcal{Q}(\mathcal{L}_{\theta'})$. The pairs $(\mathcal{L}_{\theta}, Q)$ and $(\mathcal{L}_{\theta'}, Q')$ are called G^{\sim} -equivalent if there exists a $g \in G^{\sim}$ such that g transforms the equation \mathcal{L}_{θ} to the equation $\mathcal{L}_{\theta'}$, and $Q' \sim g_* Q$.

The classification of reduction operators with respect to G^{\sim} will be understood as the classification in P with respect to this equivalence relation, a problem which can be investigated similar to the usual group classification in classes of differential equations. Namely, we construct firstly the reduction operators that are defined for all values of θ . Then we classify, with respect to G^{\sim} , the values of θ for which the equation \mathcal{L}_{θ} admits additional reduction operators.

In what follows we use above-mentioned method to investigate reduction operators of the (1+1)-dimensional variable coefficient nonlinear wave equations (3). For convenient, we rewrite it as the form

$$f(x)u_{tt} - (u^n u_x)_x - h(x)u^m = 0. \quad (9)$$

3.1 Singular reduction operators

Using the procedure given by Popovych et al. in [24], we can obtain the following assertion.

Proposition 1. *A vector field $Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$ is singular for the differential function $L = f(x)u_{tt} - (u^n u_x)_x - h(x)u^m$ if and only if $\xi^2 f(x) = \tau^2 u^n$.*

Proof. Suppose that $\tau \neq 0$. According to the characteristic equation $\tau u_t + \xi u_x - \eta = 0$, we can get

$$\begin{aligned} u_t &= \frac{\eta}{\tau} - \frac{\xi}{\tau} u_x, \\ u_{tt} &= \left(\frac{\eta}{\tau}\right)_t - \left(\frac{\xi}{\tau}\right)_t u_x + \left[\left(\frac{\eta}{\tau}\right)_u - \left(\frac{\xi}{\tau}\right)_u u_x\right] \left(\frac{\eta}{\tau} - \frac{\xi}{\tau} u_x\right) - \left(\frac{\xi}{\tau}\right) \left[\left(\frac{\eta}{\tau}\right)_x + \left(\frac{\eta}{\tau}\right)_u u_x - \left(\frac{\xi}{\tau}\right)_x u_x \right. \\ &\quad \left. - \left(\frac{\xi}{\tau}\right)_u u_x^2 - \left(\frac{\xi}{\tau}\right) u_{xx}\right]. \end{aligned}$$

Substituting the formulas of u_{tt} from above formulae into L , we obtain a differential function

$$\begin{aligned}\tilde{L} = & [f(x)(\frac{\xi}{\tau})^2 - u^n]u_{xx} + f(x)\left\{(\frac{\eta}{\tau})_t - (\frac{\xi}{\tau})_t u_x + [(\frac{\eta}{\tau})_u - (\frac{\xi}{\tau})_u u_x](\frac{\eta}{\tau} - \frac{\xi}{\tau} u_x) \right. \\ & \left. - (\frac{\xi}{\tau})[(\frac{\eta}{\tau})_x + (\frac{\eta}{\tau})_u u_x - (\frac{\xi}{\tau})_x u_x - (\frac{\xi}{\tau})_u u_x^2]\right\} - nu^{n-1}u_x^2 - hu^m u_x.\end{aligned}$$

According to the definition 3 of singular vector field, we have $\text{ord } \tilde{L} < 2$ if and only if $f(x)(\frac{\xi}{\tau})^2 - u^n = 0$. \square

Therefore, for any f, h, n and n with $fu^n > 0$ the differential function $L = f(x)u_{tt} - (u^n u_x)_x - h(x)u^m$ possesses exactly two set of singular vector fields in the reduced form, namely, $S = \{\partial_t + \sqrt{u^n/f}\partial_x + \hat{\eta}\partial_u\}$ and $S^* = \{\partial_t - \sqrt{u^n/f}\partial_x + \hat{\eta}\partial_u\}$, where $\hat{\eta} = \frac{\eta}{\tau}$. Any singular vector field of L is equivalent to one of the above fields. The singular sets are mapped to each other by alternating the sign of x and hence one of them can be excluded from the consideration.

Proposition 2. *For any variable coefficient nonlinear wave equations in the form (9) the differential function $L = f(x)u_{tt} - (u^n u_x)_x - h(x)u^m$ possesses exactly one set of singular vector fields in the reduced form, namely, $S = \{\partial_t + \sqrt{u^n/f}\partial_x + \hat{\eta}\partial_u\}$.*

Thus taking into account the conditional invariance criterion for an equation from class (9) and the operator $\partial_t + \sqrt{u^n/f}\partial_x + \eta\partial_u$, we can get

Theorem 6. *Every singular reduction operator of an equation from class (9) is equivalent to an operator of the form*

$$Q = \partial_t + \sqrt{u^n/f(x)}\partial_x + \eta(t, x, u)\partial_u,$$

where the real-valued function $\eta(t, x, u)$ satisfies the determining equations

$$\begin{aligned}& (-1/2nhu^{m-1} - 2f\eta_{tu} - 2f\eta\eta_{uu})\sqrt{u^n/f} + (3/4f_x^2/f - 1/2f_{xx})(u^n/f)^{3/2} \\ & + (-1/4n^2\eta^2u^{n-2} - n\eta\eta_u u^{n-1} + 1/2n\eta^2u^{n-2})(u^n/f)^{-1/2} - n\eta_x u^{n-1} \\ & - 1/2n\eta u^{n-1}f_x/f - 2\eta_{xu}u^n = 0, \\ & -\eta_{xx}u^n - n\eta\eta_x u^{n-1}(u^n/f)^{-1/2} + h\eta_u u^m + 2f\eta\eta_{tu} + f\eta^2\eta_{uu} \\ & - mh\eta u^{m-1} + f\eta_{tt} + (hf_x/f - h_x)u^m\sqrt{u^n/f} = 0.\end{aligned}\tag{10}$$

3.2 Regular reduction operators

The above investigation of singular reduction operators of nonlinear wave equation of the form (9) shows that for these equations the regular case of the natural partition of the corresponding sets of reduction operators is singled out by the conditions $\xi \neq \pm\sqrt{u^n/f}\tau$. After factorization with respect to the equivalence relation of vector fields, we obtain the defining conditions of regular subset of reduction operator: $\tau = 1, \xi \neq \pm\sqrt{u^n/f}$. Hence we have

Proposition 3. *For any variable coefficient nonlinear wave equations in the form (9) the differential function $L = f(x)u_{tt} - (u^n u_x)_x - h(x)u^m$ possesses exactly one set of regular vector fields in the reduced form, namely, $S = \{\partial_t + \xi\partial_x + \hat{\eta}\partial_u\}$ with $\hat{\xi} \neq \pm\sqrt{u^n/f}$.*

Consider the conditional invariance criterion for an equation from class (9) and the operator $\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$ with $\xi(t, x, u) \neq \pm\sqrt{u^n/f}$, we can get the following determining equations for the coefficients ξ and η :

$$\begin{aligned}& \xi_u = 0, \quad 2f\xi_t - n\eta u^{n-1} + (2\xi_x + \xi f_x/f)u^n = 0, \\ & (2n\xi_x - n\eta_u + n\xi f_x/f)u^{n-1} + (n\eta - n^2\eta)u^{n-2} - \eta_{uu}u^n + f\xi^2\eta_{uu} = 0, \\ & 2f\xi_t\xi_x - 2f\xi_t\eta_u - 2n\eta_x u^{n-1} - f\xi_{tt} - 2f\xi\eta\eta_{uu} - 2f\xi\eta_{tu} + (\xi_{xx} - 2\eta_{xu})u^n = 0, \\ & (\xi h f_x/f - \xi h_x + h\eta_u)u^m + f\eta^2\eta_{uu} + 2f\eta\eta_{tu} - 2f\xi_t\eta_x - \eta_{xx}u^n + f\eta_{tt} - mh\eta u^{m-1} = 0.\end{aligned}\tag{11}$$

From the first two equations of system (11), we have

$$\xi = \xi(t, x), \quad \eta = \frac{1}{n} 2f \xi_t u^{1-n} + \frac{1}{n} (2\xi_x + \xi \frac{f_x}{f}) u.$$

Substituting these expression into the last three equations of system (11), we have the following assertion.

Theorem 7. *Every regular reduction operator of an equation from class (9) is equivalent to an operator of the form*

$$Q = \partial_t + \xi(t, x) \partial_x + \eta(t, x, u) \partial_u \quad \text{with} \quad \eta(t, x, u) = \frac{1}{n} 2f \xi_t u^{1-n} + \frac{1}{n} (2\xi_x + \xi \frac{f_x}{f}) u, \quad (12)$$

where the real-valued function $\xi(t, x)$ satisfies the overdetermined system of partial differential equations

$$\begin{aligned} 2(1-n)f\xi_t &= 0, \quad 2(1-n)f^2\xi^2\xi_t = 0, \quad 8(1-n)f^3\xi\xi_t^2 = 0, \\ 4(1-n)[(f\xi_t)^2 - f^2\xi\xi_t(2\xi_x + \xi \frac{f_x}{f}) + f^2\xi\xi_{tt}] &= 0, \\ \xi_{xx} - 2(1 + \frac{1}{n})(2\xi_x + \xi \frac{f_x}{f})_x &= 0, \\ 2f\xi_t\xi_x - \frac{2}{n}f\xi_t(2\xi_x + \xi \frac{f_x}{f}) - 4(f\xi_t)_x - f\xi_{tt} - \frac{2}{n}f\xi(2\xi_x + \xi \frac{f_x}{f})_t - \frac{4}{n}(1-n)(f\xi_t)_x &= 0, \\ (\xi h \frac{f_x}{f} - \xi h_x + \frac{1}{n}h(1-m)(2\xi_x + \xi \frac{f_x}{f}))u^m + \frac{2}{n}(1-n-m)f h \xi_t u^{m-n} & \\ - \frac{8}{n^2}(1-n)f^4(\xi_t)^3 u^{1-3n} + \frac{8}{n^2}(1-n)[f^2\xi_t(f\xi_t)_t - f^3\xi_t^2(2\xi_x + \xi \frac{f_x}{f})]u^{1-2n} & \\ + [\frac{2}{n}f(f\xi_t)_{tt} - \frac{4}{n}f\xi_t(f\xi_t)_x + \frac{4}{n^2}f^2\xi_t(2\xi_x + \xi \frac{f_x}{f})_t - \frac{2}{n^2}(1-n)f^2\xi_t(2\xi_x + \xi \frac{f_x}{f})^2 & \\ + \frac{4}{n^2}(1-n)f(f\xi_t)_t(2\xi_x + \xi \frac{f_x}{f})]u^{1-n} + [\frac{2}{n^2}f(2\xi_x + \xi \frac{f_x}{f})(2\xi_x + \xi \frac{f_x}{f})_t & \\ + \frac{1}{n}f(2\xi_x + \xi \frac{f_x}{f})_{tt} - \frac{2}{n}f\xi_t(2\xi_x + \xi \frac{f_x}{f})_x - \frac{2}{n}(f\xi_t)_{xx}]u - \frac{1}{n}(2\xi_x + \xi \frac{f_x}{f})_{xx}u^{n+1} &= 0. \end{aligned} \quad (13)$$

Solving the above system with respect to the coefficient functions ξ, f and h under the equivalence group G_1^\sim , we can get a classification of regular reduction operator for the class (9). It is easy to know that some of the regular reduction operator are equivalent to Lie symmetry operators, while some of are nontrivial. Below, we give a detailed investigations for these cases.

In fact, the first three equations of system (13) implies there are two cases should be considered: $n \neq 1$ or not. (It should be noted that $\xi = 0$ should be exclude from the consideration because it leads to $\eta = 0$).

Case 1: $n \neq 1$. In this case, we have $\xi_t = 0$. Thus system (13) can be reduced to

$$\begin{aligned} (3n+4)\xi_{xx} + 2(n+1)(\xi \frac{f_x}{f})_x &= 0, \\ (\xi h \frac{f_x}{f} - \xi h_x + \frac{1}{n}(1-m)h(2\xi_x + \xi \frac{f_x}{f}))u^m - \frac{1}{n}(2\xi_x + \xi \frac{f_x}{f})_{xx}u^{n+1} &= 0. \end{aligned} \quad (14)$$

Thus, there are two cases should be considered: $m \neq n+1$ or not.

Case 1.1: For $m \neq n+1$, from the second equation of (14) we obtain

$$\xi h \frac{f_x}{f} - \xi h_x + \frac{1}{n}(1-m)h(2\xi_x + \xi \frac{f_x}{f}) = 0, \quad (2\xi_x + \xi \frac{f_x}{f})_{xx} = 0. \quad (15)$$

The first equation of (14) suggests that $(3n+4)\xi_x + 2(n+1)\xi \frac{f_x}{f}$ is independent of the variable x , so there exists a constant r such that $(3n+4)\xi_x + 2(n+1)\xi \frac{f_x}{f} = nr$. The second equation of (15) suggests that there exist two constants a and b such that $2\xi_x + \xi \frac{f_x}{f} = nax + nb$. By solving the last two equations we obtain

$$\xi_x = 2(n+1)(ax+b) - r, \quad \xi \frac{f_x}{f} = 2r - (3n+4)(ax+b),$$

which together with the first equation of (15) imply

$$\begin{aligned}\xi &= a(n+1)x^2 + [2b(n+1) - r]x + s, \\ f(x) &= \exp\left(\int \frac{2r - (3n+4)(ax+b)}{a(n+1)x^2 + [2b(n+1) - r]x + s} dx\right), \\ h(x) &= \exp\left(\int \frac{2r - (m+3n+3)(ax+b)}{a(n+1)x^2 + [2b(n+1) - r]x + s} dx\right),\end{aligned}\tag{16}$$

where a, b, r, s are arbitrary constants. Thus, the corresponding regular reduction operator has the form

$$Q = \partial_t + [a(n+1)x^2 + (2b(n+1) - r)x + s]\partial_x + (ax+b)u\partial_u,$$

which is equivalent to Lie symmetry operator.

Case 1.2: $m = n + 1$. In this case, system (14) can be rewritten as

$$\begin{aligned}(3n+4)\xi_{xx} + 2(n+1)(\xi \frac{f_x}{f})_x &= 0, \\ \xi h_x + 2h\xi_x + \frac{1}{n}(2\xi_x + \xi \frac{f_x}{f})_{xx} &= 0.\end{aligned}\tag{17}$$

Integrating these two equations with respect to functions $f(x)$ and $g(x)$, we can obtain

$$f(x) = |\xi|^{-\frac{3n+4}{2n+2}} \exp\left(r \int \frac{1}{\xi} dx\right), \quad h(x) = \frac{\xi_x^2 - 2\xi\xi_{xx} - p}{4(n+1)\xi^2}.$$

where p, r are arbitrary constants, ξ is an arbitrary smooth function and $n \neq -1$. In addition, $\eta = \frac{1}{n}(2\xi_x + \xi \frac{f_x}{f})u = (\frac{r}{n} + \frac{\xi_x}{2n+2})u$. Thus, we have a nontrivial regular reduction operator

$$Q = \partial_t + \xi(x)\partial_x + \left[\left(\frac{r}{n} + \frac{\xi_x}{2n+2}\right)u\right]\partial_u, \quad n \neq -1.\tag{18}$$

It should be noted that for $n = -1$ the reduction operator is also equivalent to Lie symmetry operator.

Case 2: $n = 1$. In this case, we have $\eta = 2f\xi_t + (2\xi_x + \xi \frac{f_x}{f})u$. Thus, system (13) can be reduced to

$$\begin{aligned}(7\xi_x + 4\xi \frac{f_x}{f})_x &= 0, \quad 2[\xi_x + 2(\xi + 1)\frac{f_x}{f}]\xi_t + 4(\xi + 1)\xi_{tx} + \xi_{tt} = 0, \\ [2f(2\xi_x + \xi \frac{f_x}{f})(2\xi_x + \xi \frac{f_x}{f})_t + f(2\xi_x + \xi \frac{f_x}{f})_{tt} - 2f\xi_t(2\xi_x + \xi \frac{f_x}{f})_x - 2(f\xi_t)_{xx}]u \\ - (2\xi_x + \xi \frac{f_x}{f})_{xx}u^2 + 2f^2(2\xi_t\xi_{tx} + \xi_{ttt}) + [\xi h \frac{f_x}{f} - \xi h_x + (1-m)h(2\xi_x + \xi \frac{f_x}{f})]u^m \\ - 2mhf\xi_tu^{m-1} &= 0.\end{aligned}\tag{19}$$

After some brief analysis, we find that there are five cases should be considered.

Case 2.1: $m = 0$. In this case, the third equation of (19) implies

$$\begin{aligned}2h\xi \frac{f_x}{f} - h_x\xi + 2h\xi_x + 4f^2\xi_t\xi_{tx} + 2f^2\xi_{ttt} &= 0, \\ 2f(2\xi_x + \xi \frac{f_x}{f})(2\xi_x + \xi \frac{f_x}{f})_t + f(2\xi_x + \xi \frac{f_x}{f})_{tt} - 2f\xi_t(2\xi_x + \xi \frac{f_x}{f})_x - 2(f\xi_t)_{xx} &= 0, \\ (2\xi_x + \xi \frac{f_x}{f})_{xx} &= 0.\end{aligned}\tag{20}$$

From the last equation of system (20) we can know that there exist two functions $a(t)$ and $b(t)$ such that $2\xi_x + \xi \frac{f_x}{f} = a(t)x + b(t)$. On the other hand, the first equation of (19) implies there exists a function $c(t)$ such that $7\xi_x + 4\xi \frac{f_x}{f} = c(t)$. Solving the last two equations gives

$$\xi_x = 4a(t)x + 4b(t) - c(t), \quad \xi \frac{f_x}{f} = -7a(t)x - 7b(t) + 2c(t),$$

from which we can get

$$\begin{aligned}\xi &= 2a(t)x^2 + 4b(t)x - c(t)x + d(t), \\ f(x) &= \exp\left(\int \frac{-7a(t)x - 7b(t) + 2c(t)}{2a(t)x^2 + 4b(t)x - c(t)x + d(t)} dx\right).\end{aligned}\quad (21)$$

where $d(t)$ is an arbitrary functions. Since $\frac{f_x}{f}$ is independent of t , we see that

$$\left[\frac{-7a(t)x - 7b(t) + 2c(t)}{2a(t)x^2 + 4b(t)x - c(t)x + d(t)}\right]_t = 0,$$

which leads to

$$\begin{cases} 14[a(t)b'(t) - a'(t)b(t)] + 3[a'(t)c(t) - a(t)c'(t)] = 0, \\ [b(t)c'(t) - b'(t)c(t)] + 7[a(t)d'(t) - a'(t)d(t)] = 0, \\ 2[c'(t)d(t) - c(t)d'(t)] + 7[b(t)d'(t) - b'(t)d(t)] = 0. \end{cases}\quad (22)$$

Now, we multiply both sides of the second equation of (19) by ξ and substitute (21) into it, then simplify the equation and compare the coefficient of $x^i (i = 0, 1, \dots, 5)$ to obtain

$$\begin{cases} a'(t) = 0, \\ a^2[-4b'(t) + c'(t)] = 0, \\ a[8c(t)c'(t) + 20ad'(t) + c''(t) - 4b''(t) - 32c(t)b'(t) + 112b(t)b'(t) - 28b(t)c'(t)] = 0, \\ -4c''(t)b(t) + 32ac(t)d'(t) - 80ab'(t) + 20ac'(t) - 4b''(t)c(t) - 120ab(t)d'(t) \\ + c''(t)c(t) + 2d''(t)a + 2c^2(t)c'(t) - 12b(t)c(t)c'(t) + 16b(t)^2c'(t) - 8c^2(t)b'(t) \\ + 48b(t)c(t)b'(t) + 16b''(t)b(t) - 64b^2(t)b'(t) - 16ad(t)b'(t) + 4ad(t)c'(t) = 0, \\ [-20ad'(t) + 48b(t)b'(t) + 2c(t)c'(t) - 12b(t)c'(t) - 8c(t)b'(t) + 4b''(t) - c''(t)]d(t) \\ - 80b^2(t)d'(t) + 4d''(t)b(t) - d''(t)c(t) - 4c(t)c'(t) + 44b(t)c(t)d'(t) - 28ad'(t) \\ - 48b(t)b'(t) - 6c^2(t)d'(t) + 12b(t)c'(t) + 16c(t)b'(t) = 0, \\ [16b'(t) - 4c'(t)]d^2(t) + [-20b(t)d'(t) - 4c'(t) + 6c(t)d'(t) + d''(t) + 16b'(t)]d(t) \\ + 8c(t)d'(t) - 28b(t)d'(t) = 0. \end{cases}\quad (23)$$

Note that ξ is assumed not to be identical with zero, after some simple but lengthy computations, we find that systems (22) and (23) can be reduced to:

$$a'(t) = 0, \quad b'(t) = 0, \quad c'(t) = 0, \quad d'(t) = 0 \quad (24)$$

or

$$a = 0, \quad c(t) = 4b(t), \quad d(t) = qb(t), \quad qb''(t) + 4qb(t)b'(t) + 4b'(t) = 0 \quad (25)$$

or

$$a = 0, \quad 2c(t) = 7b(t), \quad b''(t) = -3b(t)b'(t), \quad b(t)d'(t) + 2b'(t)(d(t) + 1) + d''(t) = 0 \quad (26)$$

or

$$a = 0, \quad c(t) = 3b(t), \quad d = qb(t), \quad b''(t) + 2b(t)b'(t) = 0, \quad (27)$$

where q is an arbitrary constant.

Case 2.1a: If system (24) is satisfied, then $\xi_t = 0$, the second equation of (20) is an identity. The expression (21) can be rewritten as

$$\begin{aligned}\xi &= 2ax^2 + 4bx - cx + d, \\ f(x) &= \exp\left(\int \frac{-7ax - 7b + 2c}{2ax^2 + 4bx - cx + d} dx\right),\end{aligned}\tag{28}$$

where a, b, c and d are arbitrary constants. The first equation of (20) is reduced to

$$\frac{h_x}{h} = \frac{2(\xi \frac{f_x}{f} + \xi_x)}{\xi}.$$

Substitute the expression of ξ and $f(x)$ into it and integrate both sides to obtain

$$h(x) = \exp\left(\int \frac{-6ax - 6b + 2c}{2ax^2 + 4bx - cx + d} dx\right).$$

In addition, $\eta = 2f\xi_t + (2\xi_x + \xi \frac{f_x}{f})u = (ax + b)u$. Therefore, we have

$$\begin{aligned}\xi &= 2ax^2 + 4bx - cx + d, \\ \eta &= (ax + b)u, \\ f(x) &= \exp\left(\int \frac{-7ax - 7b + 2c}{2ax^2 + 4bx - cx + d} dx\right), \\ h(x) &= \exp\left(\int \frac{-6ax - 6b + 2c}{2ax^2 + 4bx - cx + d} dx\right).\end{aligned}\tag{29}$$

where a, b, c, d are arbitrary constants. Thus, the corresponding regular reduction operator has the form

$$Q = \partial_t + (2ax^2 + 4bx - cx + d)\partial_x + (ax + b)u\partial_u,$$

which is equivalent to Lie symmetry operator.

Case 2.1b: If system (25) is satisfied, then the expression (21) can be rewritten as

$$\xi = qb(t), \quad f(x) = \exp\left(\frac{x}{q}\right).\tag{30}$$

Hence, $\xi_x = 0$, $k = b(t)$. Substituting these formulaes into the second equation of (20) we obtain

$$qb''(t) + 2qb(t)b'(t) - 2b'(t) = 0.$$

Combine it with the fourth equation of (25) to get $b'(t) = 0$. Hence $a(t), b(t), c(t), d(t)$ satisfy system (24), and the solution is included in the case 2.1a.

Case 2.1c: If system (26) is satisfied, then the expression (21) can be rewritten as

$$\xi = \frac{1}{2}b(t)x + d(t), \quad f(x) = 1 \pmod{G_1^\sim}.\tag{31}$$

Substitute it into the second equation of (20) to obtain $2b(t)b'(t) + b''(t) = 0$. Combine it with the third equation of (26) to get $b'(t) = 0$. Substitute it into the fourth equation of (26) to obtain $bd'(t) + d''(t) = 0$, which implies $d(t) = \gamma_1 e^{-bt} + \gamma_0$, where γ_1 and γ_0 are arbitrary constants. Therefore $\xi = \frac{b}{2}x + \gamma_1 e^{-bt} + \gamma_0$. Substitute it into the first equation of (20) to obtain

$$2\gamma_1(h_x + 2b^3)e^{-bt} + bxh_x + 2\gamma_0h_x - 2bh = 0.$$

Since h, γ_1, γ_0 are independent of t , the preceding equation suggests that

$$2\gamma_1(h_x + 2b^3) = 0, \quad bxh_x + 2\gamma_0h_x - 2bh = 0,$$

which leads to $b = 0$. Therefore, we have

$$\xi = d_1t + d_0, \quad \eta = 2d_0, \quad f = 1, \quad h = h_0 \pmod{G^\sim},$$

where d_1, d_0, h_0 are constants. Thus, the corresponding regular reduction operator has the form

$$Q = \partial_t + (d_1t + d_0)\partial_x + 2d_1u\partial_u,$$

which is equivalent to Lie symmetry operator.

Case 2.1d: If system (27) is satisfied, then the expression (21) can be rewritten as

$$\xi = b(t)(x + q), \quad f(x) = \frac{1}{x + q} \pmod{G_1^\sim}. \quad (32)$$

Substitute it into the second equation of (20) to obtain $2b(t)b'(t) + b''(t) = 0$, which is equivalent to the fourth equation of (27), and which leads to $2b'^2(t) + b'''(t) = -2b(t)b''(t)$. Substitute (32) into the first equation of (20) to obtain

$$h_x = 2f^2[2b'^2(t) + b'''(t)]/b(t) = 2f^2[-2b(t)b''(t)]/b(t) = -4f^2b''(t).$$

Since h and f are independent of t , there is a constant r such that $b''(t) = r$. It follows that there exist constants s and w such that $b(t) = rt^2/2 + st + w$. Substitute it into the fourth equation of (27) to obtain

$$r^2t^3 + 3rst^2 + 2(wr + s^2)t + 2ws + r = 0.$$

Then $r = 0$ and $s = 0$. Hence $b(t) = w$, $\xi_t = 0$, the solution is included in the case 2.1a.

Case 2.2: When $m = 1$, the third equation of (19) implies

$$\begin{aligned} & \xi h \frac{f_x}{f} - \xi h_x + f(2\xi_x + \xi \frac{f_x}{f})_{tt} + 2f(2\xi_x + \xi \frac{f_x}{f})(2\xi_x + \xi \frac{f_x}{f})_t \\ & - 2f\xi_t(2\xi_x + \xi \frac{f_x}{f})_x - 2(f\xi_t)_{xx} = 0, \\ & 2f\xi_t\xi_{tx} + f\xi_{ttt} - h\xi_t = 0, \\ & (2\xi_x + \xi \frac{f_x}{f})_{xx} = 0. \end{aligned} \quad (33)$$

Similar to the case of $m = 0$, from the third equation of (33) and the first two equations of (19) we get the expression of ξ and $f(x)$ as stated in (21), where $a(t), b(t), c(t), d(t)$ satisfy the condition (24) or (25) or (26) or (27).

Case 2.2a: If system (24) is satisfied, then $\xi_t = 0$, the second equation of (33) is an identity. The expression (21) can be rewritten as (28). The first equation of (33) is reduced to $h_x/h = f_x/f$, which leads to $h(x) = \epsilon f(x)$ ($\epsilon = \pm 1$) $\pmod{G_1^\sim}$. In addition, $\eta = 2f\xi_t + (2\xi_x + \xi \frac{f_x}{f})u = (ax + b)u$. Thus, we have

$$\begin{aligned} & \xi = 2ax^2 + 4bx - cx + d, \\ & \eta = (ax + b)u, \\ & f(x) = \exp\left(\int \frac{-7ax - 7b + 2c}{2ax^2 + 4bx - cx + d} dx\right), \\ & h(x) = \epsilon f(x). \end{aligned} \quad (34)$$

where a, b, c, d are arbitrary constants and $\epsilon = \pm 1$. Thus, the corresponding regular reduction operator has the form

$$Q = \partial_t + (2ax^2 + 4bx - cx + d)\partial_x + (ax + b)u\partial_u,$$

which is equivalent to Lie symmetry operator.

Case 2.2b: If system (25) is satisfied, then the expression (21) can be rewritten as (30). Hence, $\xi_x = 0$, $k = b(t)$. If $b'(t) = 0$, then $a(t), b(t), c(t), d(t)$ satisfy system (24), and the solution is included in the case 2.2a. We suppose that $b'(t) \neq 0$. From the second equation of (33) we see that $h = f\xi_{ttt}/\xi_t$. Substitute it into the first equation of (33) to get $fk_{tt} + 2fkk_t - 2f_{xx}\xi_t = 0$. Further it can be reduced to $qb''(t) + 2qb(t)b'(t) - 2b'(t) = 0$. Combine it with the fourth equation of (25) to get $b(t) = -3/q$ which is contradict to the hypothesis $b'(t) \neq 0$.

Case 2.2c: If system (26) is satisfied, then the expression (21) can be rewritten as (31). If $b'(t) = d'(t) = 0$, then $a(t), b(t), c(t), d(t)$ satisfy both systems (24) and (26), and the solution is included in the case 2.2a. We suppose that $b'^2(t) + d'^2(t) \neq 0$. Substitute (31) into the first equation of (33) to obtain

$$h_x = \frac{2[b''(t) + 2b(t)b''(t)]}{b(t)x + 2d(t)} \quad (35)$$

Substitute (31) into the second equation of (33) to obtain

$$h(x) = b'(t) + \frac{b'''(t)x + 2d'''(t)}{b'(t)x + 2d'(t)} \quad (36)$$

Then

$$h_x = \frac{2[b'''(t)d'(t) - b'(t)d'''(t)]}{[b'(t)x + 2d'(t)]^2}.$$

Substituting it into (35) yields

$$\frac{b'''(t)d'(t) - b'(t)d'''(t)}{[b'(t)x + 2d'(t)]^2} = \frac{b''(t) + 2b(t)b'(t)}{b(t)x + 2d(t)}.$$

Compare the coefficient of x^2 to obtain $b'^2(t)[b''(t) + 2b(t)b'(t)] = 0$. Substitute the third equation of (26) into it to obtain $b(t)b'^3(t) = 0$, hence $b'(t) = 0$. Thus the fourth equation of (26) can be reduced to $bd'(t) + d''(t) = 0$. Solving this linear ordinary differential equation gives $d(t) = \gamma_1 e^{-bt} + \gamma_0$, where γ_1 and γ_0 are two arbitrary constants. Therefore the expressions (31) and (36) can be rewritten as

$$\xi = \frac{1}{2}bx + \gamma_1 e^{-bt} + \gamma_0, \quad f(x) = 1, \quad h(x) = b^2 \mod G_1^\sim.$$

System (33) is verified to be true. In addition, $\eta = 2f\xi_t + (2\xi_x + \xi \frac{f_x}{f})u = bu - 2\gamma_1 be^{-bt}$. Therefore, we have

$$\begin{aligned} \xi &= \frac{1}{2}bx + \gamma_1 e^{-bt} + \gamma_0, & \eta &= bu - 2\gamma_1 be^{-bt}, \\ f(x) &= 1, & h(x) &= b^2, \end{aligned} \quad (37)$$

where b, γ_1, γ_0 are arbitrary constants. Thus, we have a nontrivial regular reduction operator

$$Q = \partial_t + \left(\frac{1}{2}bx + \gamma_1 e^{-bt} + \gamma_0\right)\partial_x + (bu - 2\gamma_1 be^{-bt})\partial_u. \quad (38)$$

Case 2.2d: If system (27) is satisfied, then the expression (21) can be rewritten as (32). Substitute it into the first equation of (33) to obtain

$$b(t)(x+q)[h+(x+q)h_x] = [b''(t) + 2b(t)b'(t)].$$

Substitute the fourth equation of (27) into it to get $b(t)(x+q)[h+(x+q)h_x] = 0$. It follows that $h(x) = r/(x+q)$, where r is a nonzero constant. Substitute it and (32) into the second equation of (33) to obtain $2b'^2(t) + b'''(t) - rb'(t) = 0$. From the fourth equation of (27), we find $b'''(t) = 4b^2(t)b'(t) - 2b'^2(t)$. Substitute it into the preceding equation to get $b'(t)[4b^2(t) - r] = 0$, which leads to $b'(t) = 0$. Then $a(t), b(t), c(t), d(t)$ satisfy system (24), and the solution is included in the case 2.2a.

Case 2.3: When $m = 2$, system (19) implies

$$\begin{aligned} (7\xi_x + 4\xi\frac{f_x}{f})_x &= 0, \\ 2(\xi_x + 2\xi\frac{f_x}{f})\xi_t + 4\xi_t\frac{f_x}{f} + 4\xi\xi_{tx} + 4\xi_{tx} + \xi_{tt} &= 0, \\ 2\xi_t\xi_{tx} + \xi_{ttt} &= 0, \\ 2f(2\xi_x + \xi\frac{f_x}{f})(2\xi_x + \xi\frac{f_x}{f})_t + f(2\xi_x + \xi\frac{f_x}{f})_{tt} - 4hf\xi_t - 2f\xi_t(2\xi_x + \xi\frac{f_x}{f})_x - 2(f\xi_t)_{xx} &= 0, \\ \xi h_x + 2h\xi_x + (2\xi_x + \xi\frac{f_x}{f})_{xx} &= 0. \end{aligned} \tag{39}$$

From the first and the last equation of system (39), we can get

$$f(x) = |\xi|^{-7/4} \exp(\alpha(t) \int \frac{dx}{\xi}), \quad h(x) = \frac{\xi_x^2 - 2\xi\xi_{xx} + q}{8\xi^2},$$

where $\alpha(t)$ is an arbitrary function, q is a constant. Substituting these expressions into the rest equations of system (39), we can see that $\xi(t, x)$ and $\alpha(t)$ satisfy the overdetermined system of partial differential equations

$$\begin{aligned} 2\xi_t\xi_{tx} + \xi_{ttt} &= 0, \\ \xi_{tt} - 3\xi_t\xi_x + 4\xi\xi_{tx} + 4\alpha\xi_t + 4\alpha_t &= 0, \\ 2\xi^2(\frac{1}{4}\xi_x + \alpha)_{tt} + 2\xi^2[(\frac{1}{4}\xi_x + \alpha)^2]_t - \xi^2\xi_t\xi_{xx} - \xi_t(\xi_x^2 - 2\xi\xi_{xx} + q) \\ - 4[(\alpha_t - \frac{3}{4}\xi_{tx})(\alpha - \frac{7}{4}\xi_x)\xi - \frac{3}{4}\xi^2\xi_{txx}] &= 0. \end{aligned} \tag{40}$$

In addition, we have

$$\eta = 2f\xi_t + (2\xi_x + \xi\frac{f_x}{f})u = 2\xi_t|\xi|^{-7/4} \exp(\alpha(t) \int \frac{dx}{\xi}) + [\frac{1}{4}\xi_x + \alpha(t)]u.$$

Thus, we have a nontrivial regular reduction operator

$$Q = \partial_t + \xi(t, x)\partial_x + \{2\xi_t|\xi|^{-7/4} \exp(\alpha(t) \int \frac{dx}{\xi}) + [\frac{1}{4}\xi_x + \alpha(t)]u\}\partial_u, \tag{41}$$

where $\xi(t, x)$ and $\alpha(t)$ satisfy the overdetermined system of partial differential equations (40).

In particular, if $\xi_t = 0$, from system (39) we can obtain

$$\begin{aligned} \xi &= \xi(x), \quad \eta = \frac{1}{4}(\xi_x + a)u, \\ f(x) &= |\xi|^{-7/4} \exp(\frac{a}{4} \int \frac{dx}{\xi}), \quad h(x) = \frac{\xi_x^2 - 2\xi\xi_{xx} + q}{8\xi^2}. \end{aligned} \tag{42}$$

where a, q are arbitrary constants. Thus, we have a nontrivial regular reduction operator

$$Q = \partial_t + \xi(x)\partial_x + (\frac{1}{4}\xi_x + a)u\partial_u, \tag{43}$$

which is equivalent to operator (18) with $n = 1$. Therefore, this special case can be included in case 1.2 and we can impose an additional constraint $\xi_t \neq 0$ on the regular reduction operator (41).

Case 2.4: When $m = 3$, the third equation of (19) implies

$$\begin{aligned} 2\xi_t \xi_{tx} + \xi_{ttt} &= 0, \\ 2f(2\xi_x + \xi \frac{f_x}{f})(2\xi_x + \xi \frac{f_x}{f})_t + f(2\xi_x + \xi \frac{f_x}{f})_{tt} - 2f\xi_t(2\xi_x + \xi \frac{f_x}{f})_x - 2(f\xi_t)_{xx} &= 0, \\ 6hf\xi_t + (2\xi_x + \xi \frac{f_x}{f})_{xx} &= 0, \\ \xi h \frac{f_x}{f} - \xi h_x - 2h(2\xi_x + \xi \frac{f_x}{f}) &= 0, \end{aligned} \quad (44)$$

the fourth equation of which can be rewritten as

$$\frac{\xi_x}{\xi} = -\frac{1}{4}\left(\frac{f_x}{f} + \frac{h_x}{h}\right).$$

Since f and h are independent of t , integrate both sides of the preceding equation to obtain $\xi = r(t)|fh|^{-1/4}$, where $r(t)$ is a function of t . Substituting it into the first equation of (44) yields the fact that $r'''(t) = q(x)r'^2(t)$, where $q(x) = -2(|fh|^{-1/4})_x$. It follows that $r'(t) = 0$ or $q'(x) = 0$, $r'''(t) = qr'^2(t)$.

Case 2.4a: If $r'(t) = 0$, then $\xi = r|fh|^{-1/4}$ ($r = \text{const}$), $\xi_t = 0$, $g_t = 0$, the second equation of (44) is an identity, so is the second equation of (19). The third equation of (44) reduces to $(2\xi_x + \xi \frac{f_x}{f})_{xx} = 0$. Combine it with the first equation of (19), and use a progress similar to the case $m = 0$ (i.e. 2.1a), we get the expression of ξ and $f(x)$ as stated in (28), where a, b, c, d are constants. From $\xi = r|fh|^{-1/4}$, we see that $h(x) = \pm \frac{1}{f} \left(\frac{r}{\xi}\right)^4$, where $r\xi > 0$. In addition, $\eta = 2f\xi_t + (2\xi_x + \xi \frac{f_x}{f})u = (ax + b)u$. Thus, we have

$$\begin{aligned} \xi &= 2ax^2 + 4bx - cx + d, \\ \eta &= (ax + b)u, \\ f(x) &= \exp\left(\int \frac{-7ax - 7b + 2c}{2ax^2 + 4bx - cx + d} dx\right), \\ h(x) &= \pm \frac{1}{f} \left(\frac{r}{\xi}\right)^4. \end{aligned} \quad (45)$$

where a, b, c, d, r are arbitrary constants. Thus, the corresponding regular reduction operator has the form

$$Q = \partial_t + (2ax^2 + 4bx - cx + d)\partial_x + (ax + b)u\partial_u,$$

which is equivalent to Lie symmetry operator.

Case 2.4b: If $q'(x) = 0$, $r'''(t) = qr'^2(t)$, then $(|fh|^{-1/4})_x = -\frac{1}{2}q$. Integration both sides of it gives $|fh|^{-1/4} = -\frac{1}{2}qx + s$, where s is a constant. Therefore $\xi = r(t)|fh|^{-1/4} = r(t)(-\frac{1}{2}qx + s)$. From the first equation of (19) we see that $(2\xi_x + \xi \frac{f_x}{f})_x = \frac{1}{4}\xi_{xx} + \frac{1}{4}(7\xi_x + 4\xi \frac{f_x}{f})_x = \frac{1}{4}\xi_{xx}$. Substituting the last two expressions into the third equation of (44), yields $r'(t) = 0$ or $\xi = 0$, which are the cases have already been discussed.

Case 2.5: When $m \neq 0, 1, 2, 3$, the third equation of (19) implies

$$\begin{aligned} \xi h \frac{f_x}{f} - \xi h_x + (1 - m)h(2\xi_x + \xi \frac{f_x}{f}) &= 0, \\ \xi_t &= 0, \\ (2\xi_x + \xi \frac{f_x}{f})_{xx} &= 0. \end{aligned} \quad (46)$$

Notice that the second equation of (46) indicates that ξ is independent of t , therefore the second equation of (19) is satisfied automatically. Similar to the case of $m = 0$, from the third equation of (46) and the first equation of (19) we get the expression of ξ and $f(x)$ as stated in (28), where a, b, c, d are constants. Substituting the expression of ξ and f into the first equation of (46) we obtain

$$h(x) = \exp \left(\int \frac{-7ax - 7b + 2c + (1-m)(ax+b)}{2ax^2 + 4bx - cx + d} dx \right).$$

In addition, $\eta = 2f\xi_t + (2\xi_x + \xi \frac{f_x}{f})u = (ax+b)u$. Therefore, we have

$$\begin{aligned} \xi &= 2ax^2 + 4bx - cx + d, \\ \eta &= (ax+b)u, \\ f(x) &= \exp \left(\int \frac{-7ax - 7b + 2c}{2ax^2 + 4bx - cx + d} dx \right), \\ h(x) &= \exp \left(\int \frac{-7ax - 7b + 2c + (1-m)(ax+b)}{2ax^2 + 4bx - cx + d} dx \right). \end{aligned} \quad (47)$$

where a, b, c, d are arbitrary constants. Thus, the corresponding regular reduction operator has the form

$$Q = \partial_t + (2ax^2 + 4bx - cx + d)\partial_x + (ax+b)u\partial_u,$$

which is equivalent to Lie symmetry operator.

From the above discussion, we can arrival at the following two theorems.

Theorem 8. *A complete list of G_1^\sim -inequivalent equations (3) having nontrivial regular reduction operator is exhausted by ones given in table 2.*

Table 2. Results of regular reduction operator classification of class (3)

N	n	m	$f(x)$	$h(x)$	Regular reduction operator Q
1	$\neq -1$	$n+1$	$ \xi ^{-\frac{3n+4}{2n+2}} \exp(r \int \frac{1}{\xi} dx)$	$\frac{\xi_x^2 - 2\xi\xi_{xx} - p}{4(n+1)\xi^2}$	$\partial_t + \xi(x)\partial_x + (\frac{r}{n} + \frac{\xi_x}{2n+2})u\partial_u$
2	1	1	1	b^2	$\partial_t + (\frac{1}{2}bx + \gamma_1 e^{-bt} + \gamma_0)\partial_x + (bu - 2\gamma_1 b e^{-bt})\partial_u$
3	1	2	$ \xi ^{-\frac{7}{4}} \exp(\alpha(t) \int \frac{dx}{\xi})$	$\frac{\xi_x^2 - 2\xi\xi_{xx} + q}{8\xi^2}$	$\partial_t + \xi(t, x)\partial_x + \{2\xi_t \xi ^{-\frac{7}{4}} \exp(\alpha(t) \int \frac{dx}{\xi}) + [\frac{1}{4}\xi_x + \alpha(t)]u\}\partial_u$

Here $r, p, b, \gamma_1, \gamma_0$ are arbitrary constants, $\xi(x)$ in case 2.1 is an arbitrary functions of the variables x , $\xi(t, x)$ and $\alpha(t)$ in case 2.3 satisfy the overdetermined system of partial differential equations (40) and $\xi_t \neq 0$.

Theorem 9. *Any reduction operator of an equations from class (3) having the form (12) with $\xi_t = 0$, $\xi_{xxx} = 0$ is equivalent to a Lie symmetry operator of this equation.*

4 Exact solutions

In this section, we construct nonclassical reduction and exact solutions for the classification models in table 2 by using the corresponding regular reduction operator. Lie reduction and exact solutions of equation from class (3) have been investigated in reference [15]. We choose case 1 in table 2 as an example to implement the reduction, the other cases can be considered in a similar way.

For the first case in table 2, the corresponding equation is

$$[|\xi|^{-\frac{3n+4}{2n+2}} \exp(r \int \frac{1}{\xi} dx)]u_{tt} - (u^n u_x)_x - \frac{\xi_x^2 - 2\xi\xi_{xx} - p}{4(n+1)\xi^2} u^{n+1} = 0, \quad (48)$$

which admit the regular reduction operator

$$Q = \partial_t + \xi(x)\partial_x + \left(\frac{r}{n} + \frac{\xi_x}{2n+2}\right)u\partial_u.$$

An ansätze constructed by this operator has the form

$$u(t, x) = \varphi(\omega) |\xi|^{\frac{1}{2n+2}} \exp\left(\frac{r}{n} \int \frac{1}{\xi} dx\right), \quad \text{where} \quad \omega = t - \int \frac{1}{\xi} dx.$$

Substituting this ansätze into equation (48) leads to the reduced ODE

$$\begin{aligned} & [(4r^2 - p)n^2 + 4(2n+1)r^2]\varphi^{n+1}(\omega) + 4n(n+1)[n\varphi''(\omega) - 2(n+1)r\varphi'(\omega)]\varphi^n(\omega) \\ & + 4n^3(n+1)\varphi'^2(\omega)\varphi^{n-1}(\omega) - 4n^2(n+1)\varphi''(\omega) = 0. \end{aligned} \quad (49)$$

Because there are higher nonlinear terms, we were not able to completely solve the above equation. Thus, we try to solve this equation under different additional constraints imposed on p and r .

We first rewrite equation (49) as

$$\begin{aligned} & 4n^2(n+1)[\varphi'(\omega)\varphi^n(\omega)]' - 4n^2(n+1)\varphi''(\omega) - 8n(n+1)^2r\varphi'(\omega)\varphi^n(\omega) \\ & + [4(n+1)^2r^2 - pn^2]\varphi^{n+1}(\omega) = 0. \end{aligned} \quad (50)$$

If we take $p = 4(1 + \frac{1}{n})^2r^2$, then the general solution of (50) can be written in the implicit form

$$\int \frac{n(\varphi^n - 1)}{2r\varphi^{n+1} + c_1} d\varphi = \omega + c_2. \quad (51)$$

Up to similarity of solutions of equation (3), the constant c_2 is inessential and can be set to equal zero by a translation of ω , which is always induced by a translation of t .

If we further set $n = 1$, the general solution (51) can be rewritten in the implicit form

$$\omega - \frac{\ln(2r\varphi^2 - c_1)}{4r} - \frac{\sqrt{2} \operatorname{arctanh}\left(\frac{\sqrt{2r\varphi}}{\sqrt{c_1r}}\right)}{2\sqrt{c_1r}} + c_2 = 0. \quad (52)$$

Thus we obtain the following solution

$$u(t, x) = \varphi(\omega) |\xi|^{\frac{1}{4}} \exp\left(r \int \frac{1}{\xi} dx\right), \quad \omega = t - \int \frac{1}{\xi} dx$$

for the equation

$$[|\xi|^{-\frac{7}{4}} \exp\left(r \int \frac{1}{\xi} dx\right)]u_{tt} - (uu_x)_x - \frac{\xi_x^2 - 2\xi\xi_{xx} - 8r^2}{8\xi^2}u^2 = 0,$$

where φ satisfy the equation (52), ξ is an arbitrary function and r is a non-zero constant.

If we further set $r = 0$, the general solution (51) can be rewritten in the implicit form

$$\frac{1}{n+1}\varphi^{n+1}(\omega) - \varphi(\omega) = c_1\omega + c_2. \quad (53)$$

Thus we obtain the following solution

$$u(t, x) = \varphi(\omega) |\xi|^{\frac{1}{2n+2}}, \quad \omega = t - \int \frac{1}{\xi} dx$$

for the equation

$$|\xi|^{-\frac{3n+4}{2n+2}}u_{tt} - (u^n u_x)_x - \frac{\xi_x^2 - 2\xi\xi_{xx}}{4(n+1)\xi^2}u^{n+1} = 0,$$

where φ satisfy the equation (53), ξ is an arbitrary function. In particular, for $n = 1$ from equation (53) we have

$$\varphi(\omega) = 1 \pm \sqrt{1 + 2(c_1\omega + c_2)}.$$

Thus we obtain an explicit solution

$$u(t, x) = [1 \pm \sqrt{1 + 2(c_1\omega + c_2)}]|\xi|^{\frac{1}{4}}, \quad \omega = t - \int \frac{1}{\xi} dx$$

for the equation

$$|\xi|^{-\frac{7}{4}}u_{tt} - (uu_x)_x - \frac{\xi_x^2 - 2\xi\xi_{xx}}{8\xi^2}u^2 = 0.$$

If we take different functions for ξ , then we can obtain a series of solutions for the corresponding equations. In order to avoid tediousness, we do not make a further discussion here.

5 Conclusion and Discussion

In this paper we have given a detailed investigation of the reduction operators of the variable coefficient nonlinear wave equations (1) (equivalently to (3)) by using the singular reduction operator and the equivalence transformation theory. A classification of regular reduction operators is performed with respect to generalized extended equivalence groups. The main results on classification for the equation (3) are collected in table 2 where we list three inequivalent cases with the corresponding regular reduction operators. Nonclassical symmetry reduction of a class nonlinear wave model (48) which are singled out the classification models are also performed. This enabled to obtain some non-Lie exact solutions which are invariant under certain conditional symmetries for the corresponding model.

The present paper is a preliminary nonclassical symmetry analysis of the class of hyperbolic type nonlinear partial differential equations (1). Therefore, further investigations of different properties such as nonclassical potential symmetries, and nonclassical potential exact solutions as well as physical application of this class of equations would be extremely interesting. These results will be reported in subsequent publications.

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